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ON SOME GRAPH MEASURES OF ROBUSTNESS IN TRAFFIC NETWORK

A BHARALI

Department of Mathematics, Dibrugarh University, Dibrugarh, Assam, India

ABSTRACT

Robustness is the ability of the traffic network to cope with exceptional changes in the demand and supply pattern due to uncertainties. It is very essential to have a measure that can quantify this robustness, because only then it will be possible for us to compare networks, to improve existing networks and to design new networks that are able to perform well when it subjects to failure. In this paper we survey some robustness measures based on different graph theoretic concepts like connectivity betweenness, centrality, Laplacian spectrum etc., and try to present a comparative study by means of some small example graphs.

KEYWORDS: Graph Invariants, Laplacian Spectrum, Traffic Networks, Robustness, Network Criticality

AMS Classification (2010): 05C12, 05C31, 05C40, 05C90

INTRODUCTION

Robustness is the ability of a network to continue performing well when it is subject to failures or attacks. As we live in a highly networked world, where vital facilities such as emergency medical attention, fire brigades depend on a large amount of networks of different kinds, thus the significance of having a robust road network becomes clear. For road authorities, a more robust road network has higher capability against the unpredicted and exceptional disturbances, which means the whole society will be less affected by these disturbances. For travelers, they can also benefit from a robust road network with less losses of time when they encounter such disturbances. But the fact is that till now the amount of studies on road network robustness are quite limited, and its concept has not been widely and well known because it is easily confused with network reliability, even to some researchers of transportation systems.

From the analysis in the sections above, road network robustness problem has the following characteristics: a physical road network with supply and demand, performance of the road network, disturbances in supply or demand, and travelers' reactions. Thus several primary requirements for a systematical analysis of road networks robustness can include the state without exceptional disturbance, modeling the disturbances to the road network, including the disturbances on the supply (capacity) and on the demand, representation of the interaction between network performance and travelers' (route) choice behavior and a valid network model (Li 20008).

In order to decide whether a given network is robust, a way to quantify network robustness needed. Intuitively robustness is all about back-up possibilities (Singer 2006), or alternative paths (Wu 2008), but it is not an easy task to capture all these concepts in one mathematical formula or parameter. Several robustness measures have been proposed during last few years (Sydney 2008). Network robustness has drawn attention by scientist with different backgrounds, like mathematics, physics, computer science, even biology (Singer 2006). As a result different approaches have been adopted to

capture the robustness properties of a network. All of these are based on the analysis of the corresponding graph of a network. This graph is obtained by considering places or intersections as vertices and the links between them as edges.

In the field of complex networks a large amount of graph measures have been studied. In the first part of this paper we consider some of the popular measures that have been proposed to quantify robustness. And later we also try to present a comparative study of these graph metrics in the context of robustness.

CLASSICAL GRAPH MEASURES

In the past decades, several graph measures have been introduced to study robustness based on some classical graph theory (Dekker 2004, Ellens 2011). In this section we treat some of these graph measures based on connectivity, distance, betweenness and clustering.

Vertex Connectivity and Edge Connectivity

The *vertex connectivity*, κ_v of an incomplete graph is the minimum number of vertices to be removed in order to disconnect the graph. The number of edges that need to be removed to disconnect the graph is called the *edge connectivity*, κ_e. It is very obvious that $\kappa_v \leq \kappa_e \leq \delta_{min}$, where δ_{min} is the minimum degree of the vertices. It is quite clear from these definitions that higher the vertex connectivity or edge connectivity of the graph of a network, the more robust the network is.

Distance

Let the *distance*, d_{ij} be the distance (the number of edges) between the vertices i and j. The maximum distance over these distances is called diameter, d_{max} and the average over all pairs is denoted by d.

$$d = \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} d_{ij} \tag{1}$$

If we consider the fact that the shorter a path, the more robust it is, then the average distance and the diameter can be considered as measures of robustness. The clear disadvantage in this consideration is that a vulnerable path can be compensated by adding back-up paths. Out of these two measures average distance is more sensible than the diameter, because an addition of an edge may not change the diameter while average distance is strictly decreasing.

Betweenness

The *betweenness* of a vertex or an edge e, is defined as sum of all the ratios between the number of shortest paths passing through e between pair of vertices i and j to the total number of shortest paths between pair of vertices i and j, where summation is taken over all possible i and j pairs. More formally,

$$b_e = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{n_{ij}(e)}{n_{ij}}, \tag{2}$$

Where $n_{ij}(e)$ is the number of shortest paths between i and j passing through e, n_{ij} is the number of shortest paths between i and j and b_e is the betweenness of e.

We know that traffic has a tendency to travel by shortest paths and the betweenness represents the importance of a vertex/edge. If the betweenness is high that means more number shortest paths are passing through that vertex/edge. So deleting vertices or edges with higher betweenness can have more impact than deleting others. Therefore betweenness can be helpful in detecting bottlenecks and give a tool to improve robustness of a network.

In order to get a measure for the robustness of the whole network we can take the maximum of all these betweenness of all vertices/edges, b_e^{max} or the average of the vertices/edges, b. The smaller this value, the more robust the network is.

Clustering Coefficient

Strogatz has proposed the concept of clustering coefficient to capture the presence of triangles in a graph, which compares the number of triangles to the number of connected triples (Strogatz 1998). The clustering coefficients, C_v of a vertex v is defined as the number of edges among neighbours of v divided by $\boldsymbol{\delta}_v(\boldsymbol{\delta}_v-1)/2$, the total possible number of edges among its neighbours, where $\boldsymbol{\delta}_v$ is the degree of the vertex v. The overall clustering coefficient of a graph, C is the average over the clustering coefficients of the vertices.

$$C = \frac{1}{n}\sum_{v\epsilon V} C_v = \frac{1}{n}\sum_{v\epsilon V} \frac{2}{\delta_v(\delta_v-1)}\, e_v = \frac{1}{n}\sum_{v\epsilon V} \frac{2}{\delta_v(\delta_v-1)}\, (A^3)_{vv}, \tag{3}$$

Where V is the set of vertices, e_v is the number of edges among neighbours of v, and A is the *adjacency matrix* (Harary 1969).

A high clustering coefficient means more number of triangles and it indicates an increase of alternative paths, results in high robustness.

SPECTRAL GRAPH MEASURES

There are different types of matrices associated to a graph. One of these matrices is the *Laplacian.* The Laplacian L is the difference of the degree matrix D and the adjacency matrix A, i.e., $L=D-A$ (Ellens 2011). Several robustness measures based on the spectra of Laplacian have been proposed. In this paper we treat three of them namely, algebraic connectivity, the number of spanning trees and network criticality.

Algebraic Connectivity

Since the Laplacian is symmetric, positive semidefinite and the rows sum up to 0, so the eigenvalues are real, no-negative and 0 is an eigenvalue (the smallest) [Mohar 1991]. Let us consider the eigenvalues as λ_i, for i=1, 2,…, n, where $|V|=n$ such that $0=\lambda_1\leq.\ \lambda_2\leq \ldots\leq\lambda_n$. The second smallest eigenvalue λ_2 of the Laplacian is called *algebraic connectivity*, first proposed by Fiedler (Fiedler 1973). The reason to believe that it is a measure of connectivity of a graph comes from the fact that the algebraic connectivity is equal to 0 iff the graph is not connected. High algebraic connectivity means high connectivity which implies high robustness. But algebraic connectivity does not strictly increase with the addition of edges. So it may not be a good idea to base the measure on a single eigenvalue of Laplacian of a graph, and people are looking for a measure which is a function of the whole Laplacian spectrum.

Number of Spanning Trees

Baras and Hovareshti suggest that the number of spanning trees as an indicator of network robustness (Baras 2009). The number of spanning trees, $\boldsymbol{\chi}$ can be written as a function of the unweighted Laplacian eigenvalues, as follows:

$$\chi = \frac{1}{n}\prod_{i=2}^{n}\lambda_i = \frac{1}{n}\det\left(L+\frac{J}{n}\right), \tag{4}$$

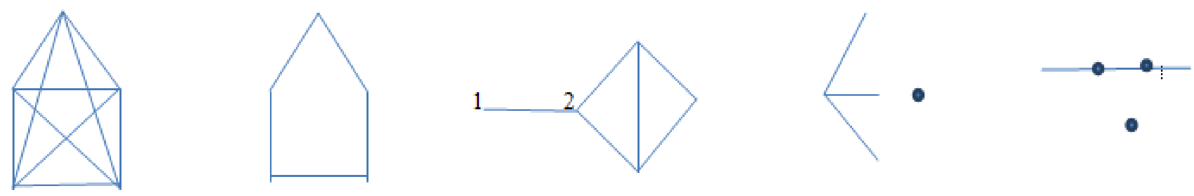
Where J is the matrix whose all entries are 1. A spanning tree is a subgraph that includes all vertices of the graph and it is a tree. So the increase of number of spanning tree implies high connectivity or in other words high robustness.

Network Criticality

To model the robustness in a network, one needs to consider the topology as well as the effect of load on the different nodes/links. In particular, the impact of a new flow on existing ones needs to be modeled. This motivates the use of betweenness metrics from graph theory. *Tizghadam* and *Garcia* come up with a new metric based on the criticality of a node/edge of a network, known as *Network Criticality* (Garcia 2010). The network criticality, τ is a strictly convex function of graph weights and it is also a non-increasing function of link weights or link capacity. Clearly, the paths with more available capacity are desired since the low available capacity paths are prone to congestion. Hence an intelligent routing plan should avoid routing the flows onto the low available capacity paths and should request for capacity increases for those paths if possible. So, smaller value of the network criticality means high robustness. Network Criticality, τ is equal to $2n\ Trace\ (L^{+})$ (Garcia 2010), where L^{+} is the Moore-Penrose inverse (Dennis 2005) of Laplacian matrix L.

COMPARISION OF ROBUSTNESS MEASURES

In this section we present a comparative study of the graph measures described in the earlier sections and discuss their ability to capture the robustness properties of a network. We star by calculating the values of all measures for the following known graphs with five vertices.



Complete Graph K_5 Cycle Graph C_5 Kite G_1 G_2=Claw U K_1 G_3= P_4 U K_1

Figure 1: Some Graphs with 5 Vertices

These graphs are ordered by decreasing number of alternative paths. Intuitively, we can say that the graphs are arranged in a decreasing order of robustness.

Table 1: The Values of the Graph Measures for the Five Graphs shown in Figure 1

	κ_v	κ_e	d_{max}	d	b_e^{max}	b	C	λ_2	χ	τ
K_5	4	4	1	1	1	1	2	5	125	8
C_5	2	2	2	1.5	3	3	0	1.382	5.002	20
G_1	1	1	3	1.5	4	2.5	1.067	.8299	39.999	20.5
G_2	0	0	∞	∞	-	-	0	0	0	22.5
G_3	0	0	∞	∞	-	-	0	0	0	25

By observing the values of the table we can conclude that the clustering coefficient and connectivity measures are poor robustness measures as they are not helpful to distinguish between more than 3 graphs out of the five graphs under consideration.

The algebraic connectivity is also successful in distinguishing the example graphs. However it does not always detect the addition of an edge, because of the fact that adding an edge does not necessarily affect the first *n-2* Laplacian eigenvalues. For example if we add an edge to star graph, the algebraic connectivity will not change. The requirement that a good robustness measure must be strictly when an edge is added, excludes a lot of measures mentioned above.

The maximum edge betweenness, average edge betweenness, average distance and number of spanning tree seem to be good measures to quantify robustness. But we can see that they are contradicting one another in case of the graphs C_5 and G_1. Now if we closely observe G_1, there is no alternative path from node 1 to node 2, which goes against robustness and but that is not the case for C_5. So we can say that the robustness is high in case of C_5 than that of G_1. And network criticality, algebraic connectivity and average edge betweenness have captured this fact.

The disadvantages of these measures are because of the fact that they are either considering connectivity or the shortest paths in a graph while for the robustness of a network the longer alternative paths are also important with the knowledge of how critical that link/path is to the changes in the topology and traffic demand of a network.. Network Criticality is a measure of robustness, which is based on betweenness and random walk that can quantify the criticality of a link/path,

CONCLUSIONS

In this survey we have studied different graph theoretic measures of robustness. These measures are based on connectivity, betweenness and some spectral graph theory. The analysis of the ten measures has shown that almost all measures are able to place some small example graphs in the same order of robustness that we would intuitively place them. But not all measures can distinguish the given graphs and sometimes they are opposing one another. There are many more graph measures of robustness have been proposed and studied in last few years, but a tool for network administrators to evaluate all the properties and improve the robustness of their networks is still a goal to be achieved. The choice of the measure of robustness depends upon the purpose of the network design. For example, if we want to model the robustness of network topologies in the face of possible node destruction then Node Connectivity is the most appropriate one, again if reducing criticality of link/path to the changes in the topology and traffic demand of a network is the aim then Network Criticality is the best one.

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